Multiscale POD (mPOD): Decomposing Tidal Water Levels at Conception Bay

Jakob Toft Pedersen Letizia Bottani

Data-Driven Methods for Computational Science and Engineering

mPOD Theory

Multiscale POD (mPOD) integrates frequency localization into the classical POD framework, separating scales with a lowpass-filter, some bandpass-filters, and a highpass-filter.

The mPOD algorithm consists of the following main steps:

1. Compute the temporal correlation matrix:
$$\mathbf{K} = \mathbf{X}^T\mathbf{X}.$$
2. Apply a multiresolution filter bank to decompose $\mathbf{K}$ into M non-overlapping frequency bands:
$$\mathbf{K} = \sum_{j=1}^{M} \mathbf{K}_j.$$
Each $\mathbf{K}_j$ contains the contribution of a specific frequency interval.
3. For each frequency band j, perform a standard POD on $\mathbf{K}_j$:
$$\mathbf{K}_j = \mathbf{V}_j\mathbf{\Lambda}_j\mathbf{V}_j^T.$$
This yields frequency-localized temporal modes $\mathbf{V}_j$ and associated eigenvalues $\mathbf{\Lambda}_j$.
4. Concatenate the temporal modes from all bands into a global basis:
$$\mathbf{V}_{\text{mPOD}} = [\mathbf{V}_1 \,|\, \mathbf{V}_2 \,|\, \dots \,|\, \mathbf{V}_M].$$
5. Reconstruct the spatial modes and amplitudes via:
$$\mathbf{U}_{\text{mPOD}} = \mathbf{X}\mathbf{V}_{\text{mPOD}}\mathbf{\Sigma}^{-1},$$
where $\mathbf{\Sigma}$ is the diagonal matrix of singular values $\sqrt{\lambda_j}$ from each $\mathbf{K}_j$.

Simple Test Case

We are using the `Modulo` Python package to compute the mPOD. To ensure both the correctness of the package and our implementation, we start by testing on a simple synthetic case.

The test case is composed of three signals; two cosine functions and a sine function modulated by a Gaussian envelope introducing temporal localization:

$$x(t) = c_1(t) + c_2(t) + s_{\text{gauss}}(t),$$

We construct a data matrix X with uniform spatial structure, i.e. all rows are equal to $x(t)$. This matrix representation allows us to apply SVD and Modulo package directly.

A standard POD applied to X captures nearly all the signal energy in a single mode. This behavior is suitable for many applications but may not be ideal when the objective is to distinguish features by their frequency content. In such cases, the mPOD is more appropriate.

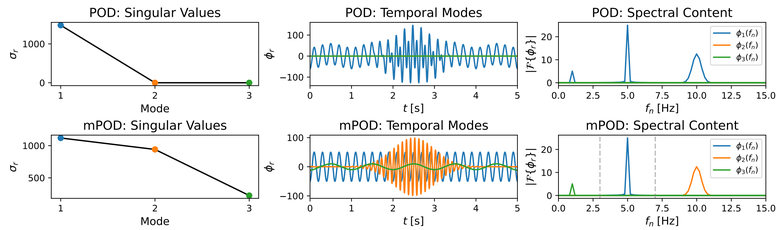


Figure 1. Comparison of POD and mPOD of $x(t) = c_1(t) + c_2(t) + s_{\text{gauss}}(t)$

Bug in Modulo

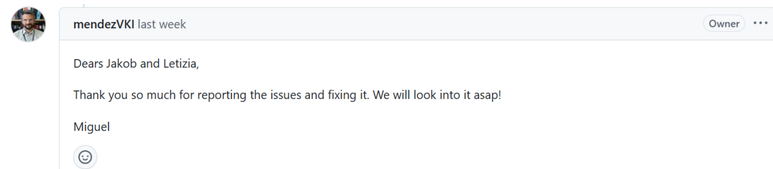
The results in Figure 1 differ from our initial findings when using the `Modulo` package to compute the mPOD.

Through repeated testing of the mPOD implementation on various synthetic cases, we observed that the second frequency filter was consistently skipped. This behavior is not in line with the framework of mPOD, and it was not present in a simple mPOD model proposed by ChatGPT.

If we had applied mPOD only on Conception Bay dataset, where the expected outcome is unknown, we would likely not have discovered the problem. This highlights the importance of validating models on well-understood test cases before applying them to complex or unknown systems.

Once the issue was identified, we traced it to an indexing bug within the `Modulo` package. After locating and correcting the bug, we reported the issue to the `Modulo` development team and shared our fixed version.

All the results presented in this work are based on the corrected version of the `Modulo` package.



Conception Bay: POD

We aim to investigate whether a meaningful separation of modes can provide insight into the tidal behavior of the water level at Conception Bay, Canada. The dataset consists of one year of measurements with a temporal resolution of 30 minutes. When applying the POD, we observe that an overwhelming 99.99% of the signal energy is captured by the first mode. This behavior is very similar to the test case shown in Figure 1. While POD performs exceptionally well as a reduced-order model in this context, the temporal mode is difficult to interpret directly.

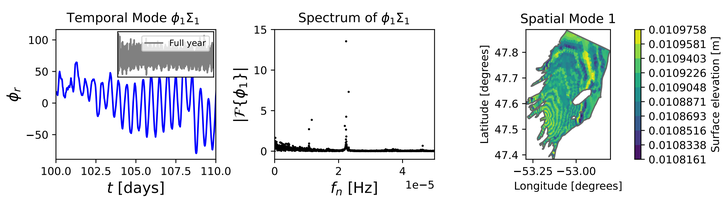


Figure 2. POD Conception Bay, zoomed in on 10 days of temporal mode

We can identify a prominent frequency of approximately 12 hours (semi-diurnal), which is characteristic of tidal dynamics. However, the temporal mode obtained from standard POD contains a broad mixture of frequencies. By applying multiscale POD (mPOD), we can separate these frequencies into narrower spectral bands, yielding more interpretable components.

Conception Bay: Spectral Decomposition

A spectral decomposition of the Conception Bay data using the FFT shows that the signal is dominated by a few distinct frequencies with much higher amplitudes than the rest. Figure 3 highlights the periods corresponding to these dominant frequencies. To isolate these components more clearly, we apply the mPOD, aiming to separate the spectral peaks into distinct temporal modes.

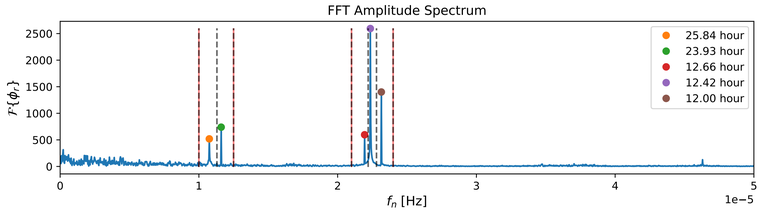


Figure 3. FFT on Conception Bay data with two examples of frequency splittings; black and red.

Based on the spectrum, different frequency splits can be used in the mPOD. Figure 3 illustrates two possible strategies: the black lines represent a fine splitting that separates all visible peaks, while the red lines indicate a coarser splitting that groups daily and half-daily tidal components. In this work, we adopt the red frequency splitting and define five distinct frequency bands for the mPOD: 1) Low frequencies. 2) Daily frequencies. 3) Frequencies between daily and half-daily. 4) Half-daily frequencies. 5) Larger frequencies.

Conception Bay: mPOD

With the red frequency splittings in Figure 3, the mPOD yields more interpretable modes compared to the POD results in Figure 2. Mode 1 and 3, shown in Figure 4, correspond to half-daily and daily oscillations, both exhibiting a modulation pattern that repeats approximately twice per month, suggesting an influence of the spring-neap tidal cycle, governed by the lunar phase.

It is worth noting that Mode 2 (not shown) captures the low-frequency content. Although this mode is not sparse in Fourier basis and therefore less interpretable within a linear model framework, it should not be disregarded, as it contains significant energy related to long-term trends.

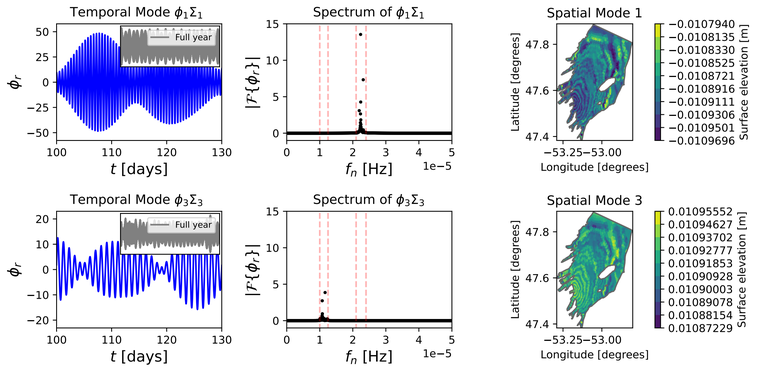


Figure 4. mPOD Conception Bay, mode 1 (half-daily periods) and mode 2 (daily periods)